STEP Solutions 2012

Mathematics
STEP 9465/9470/9475

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Hints & Solutions for Paper 9470 (STEP II) June 2012

Question 1

To be honest, the binomial expansions of $(1 \pm x)^n$, in the cases $n = 1, 2$, are used so frequently within AS- and A-levels that they should be familiar to all candidates taking STEPs. Replacing x by x^k is no great further leap.

The general term in $\left(1-x^6\right)^{-2}$ is easily seen to be $(n + 1)\, x^{6n}$ and the x^{24} term in $\left(1-x^6\right)^{-2}\left(1-x^3\right)^{-1}$ comes from $1.x^{24} + 2x^6.x^{18} + 3x^{12}.x^{12} + 4x^{18}.x^6 + 5x^{24}.1$, so that the coefficient of x^{24} is $1 + 2 + 3 + 4 + 5 = 15$, arising from a sum of triangular numbers. Thus, the coefficient of x^n is

$$\begin{cases} 0 & \text{if } n = 6k + \{1, 2, 4, 5\} \\ \tfrac{1}{2}(k+1)(k+2) & \text{if } n = 6k+3 \\ \tfrac{1}{2}(k+1)(k+2) & \text{if } n = 6k \end{cases}$$

which is most easily described without using n directly, as here.

In (ii), f(x) = $\left(1+2x^6+3x^{12}+4x^{18}+5x^{24}+\ldots\right)\left(1+x^3+x^6+x^9+\ldots\right)\left(1+x+x^2+x^3+\ldots\right)$ and the x^{24} term comes from

$$\begin{aligned}
&1.1.5x^{24} + 1.x^6.4x^{18} + 1.x^{12}.3x^{12} + 1.x^{18}.2x^6 + 1.x^{24}.1 \\
&+ x^3.x^3.4x^{18} + x^3.x^9.3x^{12} + x^3.x^{15}.2x^6 + x^3.x^{21}.1 \\
&+ x^6.1.4x^{18} + x^6.x^6.3x^{12} + x^6.x^{12}.2x^6 + x^6.x^{18}.1 \\
&+ x^9.x^3.3x^{12} + x^9.x^9.2x^6 + x^9.x^{15}.1 \\
&+ x^{12}.1.3x^{12} + x^{12}.x^6.2x^6 + x^{12}.x^{12}.1 \\
&+ x^{15}.x^3.2x^6 + x^{15}.x^9.1 \\
&+ x^{18}.1.2x^6 + x^{18}.x^6.1 \\
&+ x^{21}.x^3.1 \\
&+ x^{24}.1.1
\end{aligned}$$

giving the coefficient of x^{24} as $15 + 2 \times (10 + 6 + 3 + 1) = 55$.

However, there are lots of ways to go about doing this. For instance ...
Note that, because every non-multiple-of-3 power in bracket 3 is redundant, the x^{24} term comes from considering f(x) = $\left(1-x^6\right)^{-2}\left(1-x^3\right)^{-2}$ = $\left(1+2x^6+3x^{12}+4x^{18}+5x^{24}+\ldots\right)\left(1+2x^3+3x^6+4x^9+\ldots\right)$.
Again, every non-multiple-of-6 power in *this* 2nd bracket is also redundant, so one might consider only

$$\text{f}(x) = \left(1+3x^6+5x^{12}+7x^{18}+9x^{24}+\ldots\right)\left(1+2x^6+3x^{12}+4x^{18}+5x^{24}+\ldots\right)$$

from which the coefficient of x^{24} is simply calculated as $1 \times 5 + 3 \times 4 + 5 \times 3 + 7 \times 2 + 9 \times 1 = 55$. This result, in some form or another, gives the way of working out the coefficient of x^{6n} for any non-negative integer n. It is immediately obvious that it is $\sum_{r=0}^{n}(n+1-r)(2r+1)$ which turns out to be the same as $\sum_{r=1}^{n+1} r^2 = \tfrac{1}{6}(n+1)(n+2)(2n+3)$. The proof of this result could be by induction or direct manipulation of the standard results for Σr and Σr^2.

The coefft. of x^{25} is 55, the same as for x^{24}, since the extra x only arises from replacing 1 by x, x^3 by x^4, etc., in the first bracket's term (at each step of the working) and the coefficients are equal in each case.

In the case when $n = 11$, the coefficient of x^{66} is $12 \times 1 + 11 \times 3 + 10 \times 5 + \ldots + 2 \times 21 + 1 \times 23 = 650$.

Question 2

Firstly, $\mathrm{p}(\mathrm{q}(x))$ has degree mn.

(i) $\mathrm{Deg}[\mathrm{p}(x)] = n \Rightarrow \mathrm{Deg}[\mathrm{p}(\mathrm{p}(x))] = n^2$ & $\mathrm{Deg}[\mathrm{p}(\mathrm{p}(\mathrm{p}(x)))] = n^3$.

Deg[LHS] $\leq \max(n^3, n)$ while RHS is of degree 1. Therefore the LHS is not constant so $n = 1$ and $\mathrm{p}(x)$ is linear. Setting $\mathrm{p}(x) = ax + b \Rightarrow \mathrm{p}(\mathrm{p}(x)) = a(ax + b) + b = a^2x + (a + 1)b$ and

$$\mathrm{p}(\mathrm{p}(\mathrm{p}(x))) = a[a^2x + (a + 1)b] + b = a^3x + (a^2 + a + 1)b.$$

Then $a^3x + (a^2 + a + 1)b - 3ax - 3b + 2x \equiv 0 \Rightarrow (a^3 - 3a + 2)x + (a^2 + a - 2)b \equiv 0$

$$\Rightarrow (a - 1)(a^2 + a - 2)x + (a^2 + a - 2)b \equiv 0$$
$$\Rightarrow (a^2 + a - 2)[(a - 1)x + b] \equiv 0$$
$$\Rightarrow (a + 2)(a - 1)[(a - 1)x + b] \equiv 0$$

We have, then, that $a = -2$ or 1. In either case, b takes any (arbitrary) value and the solutions are thus

$$\mathrm{p}_1(x) = -2x + b \quad \text{and} \quad \mathrm{p}_2(x) = x + b.$$

(ii) Deg[RHS] = 4 while Deg[LHS] $\leq \max(n^2, 2n, n)$, so it follows that $n = 2$ and $\mathrm{p}(x)$ is quadratic.

Setting $\mathrm{p}(x) = ax^2 + bx + c$, we have

$$2\mathrm{p}(\mathrm{p}(x)) = 2a(ax^2 + bx + c)^2 + 2b(ax^2 + bx + c) + 2c$$
$$= 2a\left\{a^2x^4 + 2abx^3 + 2acx^2 + b^2x^2 + 2bcx + c^2\right\} + 2b(ax^2 + bx + c) + 2c$$

$$3(p(x))^2 = 3\left[a^2x^4 + 2abx^3 + \left(2ac + b^2\right)x^2 + 2bcx + c^2\right] \text{ and } -4\mathrm{p}(x) = -4ax^2 - 4bx - 4c.$$

Thus, LHS = $\left(2a^3 + 3a^2\right)x^4 + \left(4a^2b + 6ab\right)x^3 + \left(2ab^2 + 4a^2c + 2ab + 3b^2 + 6ac - 4a\right)x^2$
$+ \left(4abc + 2b^2 + 6bc - 4b\right)x + \left(2ac^2 + 2bc + 2c + 3c^2 - 4c\right)$,

while the RHS = x^4.

Equating terms gives

x^4) $\quad 2a^3 + 3a^2 - 1 = 0 \Rightarrow (a+1)^2(2a-1) \Rightarrow a = -1$ or $\frac{1}{2}$

x^3) $\quad 2ab(2a + 3) = 0 \Rightarrow b = 0$

x^2) $\quad 2a(2ac + 3c - 2) = 0 \Rightarrow c = 2$ when $a = -1$; i.e. $\mathrm{p}_1(x) = -x^2 + 2$

OR $c = \frac{1}{2}$ when $a = \frac{1}{2}$; i.e. $\mathrm{p}_2(x) = \frac{1}{2}\left(x^2 + 1\right)$.

Note that there are two sets of conditions yet to be used, so the results obtained need to be checked (visibly) for consistency:

x^1) $2b(2ac + b + 3c - 2) = 0$ checks and x^0) $c(2ac + 3c - 2) = 0$ checks also.

Question 3

It helps greatly to begin with, to note that if $t = \sqrt{x^2+1} + x$, then $\dfrac{1}{t} = \sqrt{x^2+1} - x$. These then give the result $x = \frac{1}{2}t - \frac{1}{2}t^{-1}$, from which we find $\dfrac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{2} + \frac{1}{2}t^{-2}$ and (changing the limits) $x : (0, \infty) \to t : (1, \infty)$, so that $\displaystyle\int_0^\infty \mathrm{f}\left(\sqrt{x^2+1} + x\right)\,\mathrm{d}x = \int_1^\infty \mathrm{f}(t) \times \tfrac{1}{2}\left(1 + \frac{1}{t^2}\right)\,\mathrm{d}t = \tfrac{1}{2}\int_1^\infty \mathrm{f}(x)\left(1 + \frac{1}{x^2}\right)\,\mathrm{d}x$, as required.

For the first integral, $I_1 = \displaystyle\int_0^\infty \frac{1}{\left(\sqrt{x^2+1} + x\right)^2}\,\mathrm{d}x$, we are using $\mathrm{f}(x) = \dfrac{1}{x^2}$ in the result established initially.

Then $I_1 = \tfrac{1}{2}\displaystyle\int_1^\infty \left(1 + \frac{1}{x^2}\right).\frac{1}{x^2}\,\mathrm{d}x = \tfrac{1}{2}\int_1^\infty \left(x^{-2} + x^{-4}\right)\,\mathrm{d}x = \tfrac{1}{2}\left[-\frac{1}{x} - \frac{1}{3x^3}\right]_1^\infty = \tfrac{1}{2}\left(0 + 1 + \tfrac{1}{3}\right) = \tfrac{2}{3}$.

In the case of the second integral, the substitution $x = \tan\theta \Rightarrow \mathrm{d}x = \sec^2\theta\,\mathrm{d}\theta$. Also $\sqrt{1+x^2} = \sec\theta$ and the required change of limits yields $\left(0, \tfrac{1}{2}\pi\right) \rightarrow (0, \infty)$. We then have

$$I_2 = \int_0^{\frac{1}{2}\pi} \frac{1}{(1+\sin\theta)^3}\,\mathrm{d}\theta = \int_0^{\frac{1}{2}\pi} \left(\frac{\sec\theta}{\sec\theta + \tan\theta}\right)^3 \mathrm{d}\theta \quad \text{[Note the importance of changing to sec and tan]}$$

$$= \int_0^{\frac{1}{2}\pi} \frac{\sec\theta}{(\sec\theta + \tan\theta)^3}.\sec^2\theta\,\mathrm{d}\theta = \int_0^{\infty} \frac{\sqrt{x^2+1}}{\left(\sqrt{x^2+1}+x\right)^3}\,\mathrm{d}x\,.$$

We now note, matching this up with the initial result, that we are using $\mathrm{f}(t) = \dfrac{\frac{1}{2}\left(t + \frac{1}{t}\right)}{t^3} = \dfrac{t^2+1}{2t^4}$, so that

$$I_2 = \tfrac{1}{2}\int_1^{\infty} \left(\frac{t^2+1}{t^2}\right)\left(\frac{t^2+1}{2t^4}\right)\mathrm{d}t = \tfrac{1}{4}\int_1^{\infty} \left(t^{-2} + 2t^{-4} + t^{-6}\right)\mathrm{d}t = \tfrac{1}{4}\left[-\frac{1}{t} - \frac{2}{3t^3} - \frac{1}{5t^5}\right]_1^{\infty} = \tfrac{1}{4}\left(0 + 1 + \tfrac{2}{3} + \tfrac{1}{5}\right) = \tfrac{7}{15}\,.$$

Question 4

(i) This first result is easily established: For $n, k > 1$, $n^{k+1} > n^k$ and $k + 1 > k$ so $(k+1) \times n^{k+1} > k \times n^k$

$\Rightarrow \dfrac{1}{(k+1)n^{k+1}} < \dfrac{1}{kn^k}$ (since all terms are positive).

Then $\ln\left(1 + \dfrac{1}{n}\right) = \dfrac{1}{n} - \dfrac{1}{2n^2} + \dfrac{1}{3n^3} - \dfrac{1}{4n^4} + \dfrac{1}{5n^5} - \ldots$ (a result which is valid since $0 < \dfrac{1}{n} < 1$)

$$= \frac{1}{n} - \left(\frac{1}{2n^2} - \frac{1}{3n^3}\right) - \left(\frac{1}{4n^4} - \frac{1}{5n^5}\right) - \ldots < \frac{1}{n}$$ since each bracketed term is positive, using

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the previous result. Exponentiating then gives $1 + \dfrac{1}{n} < \mathrm{e}^{\frac{1}{n}} \Rightarrow \left(1 + \dfrac{1}{n}\right)^n < \mathrm{e}$.

(ii) A bit of preliminary log. work enables us to use the $\ln(1 + x)$ result on

$$\ln\left(\frac{2y+1}{2y-1}\right) = \ln\left(1 + \frac{1}{2y}\right) - \ln\left(1 - \frac{1}{2y}\right) = \left(\frac{1}{2y} - \frac{1}{2(2y)^2} + \frac{1}{3(2y)^3} - \frac{1}{4(2y)^4} + \frac{1}{5(2y)^5} - \ldots\right)$$

$$- \left(-\frac{1}{2y} - \frac{1}{2(2y)^2} - \frac{1}{3(2y)^3} - \frac{1}{4(2y)^4} - \frac{1}{5(2y)^5} - \ldots\right)$$

$$= 2\left(\frac{1}{2y} + \frac{1}{3(2y)^3} + \frac{1}{5(2y)^5} + \ldots\right) > \frac{1}{y}$$ (since all terms after the first are positive).

Again, note that we should justify that the series is valid for $0 < \dfrac{1}{2y} < 1$ i.e. $y > \tfrac{1}{2}$ in order to justify the use of the given series. It then follows that $\ln\left(\dfrac{2y+1}{2y-1}\right)^y > 1$, and setting $y = n + \tfrac{1}{2}$ (the crucial final step)

gives $\ln\left(\dfrac{2n+2}{2n}\right)^{n+\frac{1}{2}} > 1 \Rightarrow \left(1 + \dfrac{1}{n}\right)^{n+\frac{1}{2}} > \mathrm{e}$.

(iii) This final part only required a fairly informal argument, but the details still required a little bit of care in order to avoid being too vague.

As $n \to \infty$, $\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} = \left(1+\frac{1}{n}\right)^{n} \times \left(1+\frac{1}{n}\right)^{\frac{1}{2}} \to \left(1+\frac{1}{n}\right)^{n} \times 1+ \ \to \left(1+\frac{1}{n}\right)^{n}$ from above and e is squeezed into the same limit from both above and below.

Question 5

With any curve-sketching question of this kind, it is important to grasp those features that are important and ignore those that aren't. For instance, throughout this question, the position of the y-axis is entirely immaterial: it could be drawn through any branch of the curves in question or, indeed, appear as an asymptote. So the usually key detail of the y-intercept, at $\left(0, \frac{1}{a^2-1}\right)$ in part (i), does not help decide what the function is up to. The asymptotes, turning points (clearly important in part (ii) since they are specifically requested), and any symmetries are important. The other key features to decide upon are the "short-term" (when x is small) and the "long-term" (as $x \to \pm\infty$) behaviours.

In (i), there are vertical asymptotes at $x = a - 1$ and $x = a + 1$; while the x-axis is a horizontal asymptote. There is symmetry in the line $x = a$ (a consequence of which is the maximum TP in the "middle" branch) and the "long-term" behaviour of the curve is that it ultimately resembles the graph of $y = \frac{1}{x^2}$.

(ii) Differentiating the function in (ii) gives

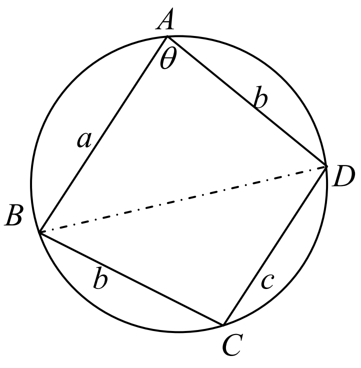
$$g'(x) = \frac{-2}{\left[(x-a)^2-1\right]^2\left[(x-b)^2-1\right]^2}\left\{(x-b)\left[(x-a)^2-1\right] + (x-a)\left[(x-b)^2-1\right]\right\}$$

and setting the numerator = 0 $\Rightarrow$ $(x-a)(x-b)\left[x-a+x-b\right] + \left[x-a+x-b\right] = 0$. Factorising yields $(2x-a-b)\left(x^2-(a+b)x+(ab-1)\right) = 0$, so that $x = \frac{1}{2}(a+b)$ or $\frac{a+b \pm \sqrt{(a+b)^2 - 4ab + 4}}{2}$.

In the first case, where $b > a + 2$ (i.e. $a + 1 < b - 1$), there are five branches of the curve, with 4 vertical asymptotes: $x = a \pm 1$ and $x = b \pm 1$. As the function changes sign as it "crosses" each asymptote, and the "long-term" behaviour is still to resemble $y = \frac{1}{x^2}$, these branches alternate above and below the x-axis, with symmetry in $x = \frac{1}{2}(a+b)$.

In the second case, where $b = a + 2$ (i.e. $a + 1 = b - 1$), the very middle section has collapsed, leaving only the four branches, but the curve is otherwise essentially unchanged from the previous case.

Question 6



A quick diagram helps here, leading to the important observation, from the GCSE geometry result "*opposite angles of a cyclic quad. add to* 180°", that $\angle BCD = 180° - \theta$. Then, using the Cosine Rule twice (and noting that $\cos(180° - \theta) = -\cos\theta$):

<u>in ΔBAD</u>: $BD^2 = a^2 + d^2 - 2ad\cos\theta$

<u>in ΔBCD</u>: $BD^2 = b^2 + c^2 + 2bc\cos\theta$

Equating for BD^2 and re-arranging gives $\cos\theta = \frac{a^2 - b^2 - c^2 + d^2}{2(ad+bc)}$

Next, the well-known formula for triangle area, $\Delta = \frac{1}{2}ab\sin C$, twice, gives $Q = \frac{1}{2}ad\sin\theta + \frac{1}{2}bc\sin\theta$, since $\sin(\pi - \theta) = \sin\theta$. Rearranging then gives $\sin\theta = \dfrac{2Q}{ad+bc}$ or $\dfrac{4Q}{2(ad+bc)}$.

Use of $\sin^2\theta + \cos^2\theta = 1 \Rightarrow \dfrac{16Q^2}{4(ad+bc)^2} + \dfrac{\left(a^2-b^2-c^2+d^2\right)^2}{4(ad+bc)^2} = 1$ and this then gives the printed result, $16Q^2 = 4(ad+bc)^2 - \left(a^2-b^2-c^2+d^2\right)^2$.

Then, $16Q^2 = \left(2ad+2bc-a^2+b^2+c^2-d^2\right)\left(2ad+2bc+a^2-b^2-c^2+d^2\right)$ by the *difference-of-two-squares* factorisation

$$= \left([b+c]^2 - [a-d]^2\right)\left([a+d]^2 - [b-c]^2\right)$$

$$= \left([b+c]-[a-d]\right)\left([b+c]+[a-d]\right)\left([a+d]-[b-c]\right)\left([a+d]+[b-c]\right)$$

using the *difference-of-two-squares* factorisation in each large bracket

$$= (b+c+d-a)(a+b+c-d)(a+c+d-b)(a+b+d-c).$$

Splitting the 16 into four 2's (one per bracket) and using $2s = a+b+c+d$

$$\Rightarrow Q^2 = \frac{(2s-2a)}{2}\frac{(2s-2b)}{2}\frac{(2s-2c)}{2}\frac{(2s-2d)}{2} = (s-a)(s-b)(s-c)(s-d).$$

Finally, for a triangle (guaranteed cyclic), letting $d \to 0$ (**Or** $s - d \to s$ **Or** let $D \to A$), we get the result known as *Heron's Formula*: $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$.

Question 7

Many of you will know that this point *G*, used here, is the centroid of the triangle, and has position vector $\mathbf{g} = \frac{1}{3}(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3)$.

Then $\overrightarrow{GX_1} = \mathbf{x}_1 - \mathbf{g} = \frac{1}{3}(2\mathbf{x}_1 - \mathbf{x}_2 - \mathbf{x}_3)$ and so $\overrightarrow{GY_1} = -\frac{1}{3}\lambda_1(2\mathbf{x}_1 - \mathbf{x}_2 - \mathbf{x}_3)$, where $\lambda_1 > 0$.

Also $\overrightarrow{OY_1} = \overrightarrow{OG} + \overrightarrow{GY_1} = \frac{1}{3}(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3) - \frac{1}{3}\lambda_1(2\mathbf{x}_1 - \mathbf{x}_2 - \mathbf{x}_3) = \frac{1}{3}\left([1-2\lambda]\mathbf{x}_1 + [1+\lambda_1](\mathbf{x}_2 + \mathbf{x}_3)\right)$, the first printed result.

The really critical observation here is that the circle centre *O*, radius 1 has equation $|\,\mathbf{x}\,|^2 = 1$ or $\mathbf{x}\,.\,\mathbf{x} = 1$, where $\mathbf{x}$ can be the p.v. of any point on the circle.

Thus, since $\overrightarrow{OY_1} \bullet \overrightarrow{OY_1} = 1$, we have

$$1 = \tfrac{1}{9}\left\{(1-2\lambda_1)^2 + 2(1+\lambda_1)^2 + 2(1-2\lambda_1)(1+\lambda_1)\mathbf{x}_1 \bullet (\mathbf{x}_2 + \mathbf{x}_3) + 2(1+\lambda_1)^2\mathbf{x}_2 \bullet \mathbf{x}_3\right\}$$

$$\Rightarrow \quad 9 = 1 - 4\lambda_1 + 4\lambda_1^{\ 2} + 2 + 4\lambda_1 + 2\lambda_1^{\ 2} + 2(1-2\lambda_1)(1+\lambda_1)\mathbf{x}_1 \bullet (\mathbf{x}_2 + \mathbf{x}_3) + 2(1+\lambda_1)^2\mathbf{x}_2 \bullet \mathbf{x}_3$$

$$\Rightarrow \quad 0 = -3(1-\lambda_1)(1+\lambda_1) + (1-2\lambda_1)(1+\lambda_1)\mathbf{x}_1 \bullet (\mathbf{x}_2 + \mathbf{x}_3) + (1+\lambda_1)^2\mathbf{x}_2 \bullet \mathbf{x}_3$$

As $\lambda_1 > 0$, $0 = -3(1-\lambda_1) + (1-2\lambda_1)\mathbf{x}_1 \bullet (\mathbf{x}_2 + \mathbf{x}_3) + (1+\lambda_1)\mathbf{x}_2 \bullet \mathbf{x}_3$

$$\Rightarrow \quad 0 = -3 + 3\lambda_1 + \left(\mathbf{x}_1 \bullet \mathbf{x}_2 + \mathbf{x}_2 \bullet \mathbf{x}_3 + \mathbf{x}_3 \bullet \mathbf{x}_1\right) + \lambda_1(\mathbf{x}_2 \bullet \mathbf{x}_3) - 2\lambda_1\left(\mathbf{x}_1 \bullet \mathbf{x}_2 + \mathbf{x}_1 \bullet \mathbf{x}_3\right)$$

$\Rightarrow \lambda_1 = \dfrac{3-(\alpha+\beta+\gamma)}{3+\alpha-2\beta-2\gamma}$, using $\alpha = \mathbf{x}_2 \bullet \mathbf{x}_3$, $\beta = \mathbf{x}_3 \bullet \mathbf{x}_1$ and $\gamma = \mathbf{x}_1 \bullet \mathbf{x}_2$.

Similarly, $\lambda_2 = \dfrac{3-(\alpha+\beta+\gamma)}{3+\beta-2\alpha-2\gamma}$ and $\lambda_3 = \dfrac{3-(\alpha+\beta+\gamma)}{3+\gamma-2\alpha-2\beta}$.

Using $\dfrac{GX_i}{GY_i} = \dfrac{1}{\lambda_i}$ $(i = 1, 2, 3)$, $\dfrac{GX_1}{GY_1} + \dfrac{GX_2}{GY_2} + \dfrac{GX_3}{GY_3} = \dfrac{1}{\lambda_1} + \dfrac{1}{\lambda_2} + \dfrac{1}{\lambda_3} = \dfrac{9 + (\alpha+\beta+\gamma) - 4(\alpha+\beta+\gamma)}{3 - (\alpha+\beta+\gamma)}$

$$= \frac{9 - 3(\alpha+\beta+\gamma)}{3 - (\alpha+\beta+\gamma)} = 3.$$

[Interestingly, this result generalises to n points on a circle: $\sum_{i=1}^{n} \dfrac{GX_i}{GY_i} = n$.]

Question 8

$\beta - \alpha > q\ (> 0) \Rightarrow \beta^2 - 2\alpha\beta + \alpha^2 > q^2 \Rightarrow \alpha^2 + \beta^2 - q^2 > 2\alpha\beta \Rightarrow \dfrac{\alpha^2 + \beta^2 - q^2}{\alpha\beta} > 2 \Rightarrow$ the opening result, $\dfrac{\alpha^2 + \beta^2 - q^2}{\alpha\beta} - 2 > 0$.

$u_{n+1} = \dfrac{u_n{}^2 - q^2}{u_{n-1}}$ etc. $\Rightarrow u_n{}^2 - u_{n+1}u_{n-1} = q^2 = u_{n+1}{}^2 - u_{n+2}u_n$ (since the result is true at all stages) and equating for $q^2 \Rightarrow u_n(u_n + u_{n+2}) = u_{n+1}(u_{n-1} + u_{n+1})$.

Now this gives $\dfrac{u_n + u_{n+2}}{u_{n+1}} = \dfrac{u_{n-1} + u_{n+1}}{u_n}$ which $\Rightarrow \dfrac{u_{n-1} + u_{n+1}}{u_n}$ is constant (independent of n). Calling this constant p gives $u_{n+1} - pu_n + u_{n-1} = 0$, as required. In order to determine p, we only need to use the fact that $p = \dfrac{u_{n-1} + u_{n+1}}{u_n}$ for all n, so we choose the first few terms to work with.

$$u_2 = \frac{\beta^2 - q^2}{\alpha} \text{ and } p = \frac{u_0 + u_2}{u_1} = \frac{\alpha + \dfrac{\beta^2 - q^2}{\alpha}}{\beta} = \frac{\alpha^2 + \beta^2 - q^2}{\alpha\beta}.$$

Alternatively, $u_2 = \gamma = \dfrac{\beta^2 - q^2}{\alpha} = p\beta - \alpha \Leftrightarrow p = \dfrac{\alpha^2 + \beta^2 - q^2}{\alpha\beta}$

and $\quad u_3 = \dfrac{\gamma^2 - q^2}{\beta} = p\gamma - \beta \Leftrightarrow p = \dfrac{\gamma^2 + \beta^2 - q^2}{\beta\gamma} = \dfrac{\left(\dfrac{\beta^2 - q^2}{\alpha}\right)^2 + \beta^2 - q^2}{\beta\left(\dfrac{\beta^2 - q^2}{\alpha}\right)}$

$$= \frac{(\beta^2 - q^2)^2 + \alpha^2(\beta^2 - q^2)}{\alpha\beta(\beta^2 - q^2)} = \frac{\beta^2 - q^2 + \alpha^2}{\alpha\beta}$$

since $\beta^2 - q^2 \neq 0$ as u_2 non-zero (given). Since p is consistent for any chosen α, β, the proof follows inductively on any two consecutive terms of the sequence.

Finally, on to the given cases.

If $\beta > \alpha + q$, $u_{n+1} - u_n = (p-1)u_n - u_{n-1} = \left(\dfrac{\beta^2 + \alpha^2 - q^2}{\alpha\beta} - 1\right)u_n - u_{n-1}$

$> (2-1)u_n - u_{n-1}$ by the initial result

$> u_n - u_{n-1}$

Hence, if $u_n - u_{n-1} > 0$ then so is $u_{n+1} - u_n$. Since $\beta > \alpha$, $u_2 - u_1 > 0$ and proof follows inductively.

If $\beta = \alpha + q$ then $p = 2$ and $u_{n+1} - u_n = u_n - u_{n-1}$ so that the sequence is an AP.
Also, $u_0 = \alpha$, $u_1 = \alpha + q$, $u_2 = \alpha + 2q$, … $\Rightarrow$ the common difference is q (and we still have a strictly increasing sequence, since $q > 0$ given).

Question 9

In the standard way, we use the constant-acceleration formulae to get

$$x = ut\cos\alpha \text{ and } y = 2h - ut\sin\alpha - \tfrac{1}{2}gt^2 .$$

When $x = a$, $t = \dfrac{a}{u\cos\alpha}$. Substituting this into the equation for y $\Rightarrow$ $y = 2h - a\tan\alpha - \dfrac{ga^2}{2u^2}\sec^2\alpha$.
As $y > h$ at this point (the ball, assuming it to be "a particle", is above the net), we get
$h - a\tan\alpha > \dfrac{ga^2}{2u^2}\sec^2\alpha \;\Rightarrow\; \dfrac{1}{u^2} < \dfrac{2(h - a\tan\alpha)}{ga^2\sec^2\alpha}$, as required.

For the next part, we set $y = 0$ in $y = 2h - ut\sin\alpha - \tfrac{1}{2}gt^2$ and solve as a quadratic in t to get
$t = \dfrac{-2u\sin\alpha + \sqrt{4u^2\sin^2\alpha + 16gh}}{2g}$ ... (the positive root is required).

Setting $x = (u\cos\alpha)t$ and noting that $x < b$, $u\cos\alpha\left(\dfrac{\sqrt{u^2\sin^2\alpha + 4gh} - u\sin\alpha}{g}\right) < b$

$\Rightarrow \sqrt{u^2\sin^2\alpha + 4gh} < \dfrac{bg}{u\cos\alpha} + u\sin\alpha$.

There are several ways to proceed from here, but this is (perhaps) the most straightforward.
Squaring $\Rightarrow$ $u^2\sin^2\alpha + 4gh < \dfrac{b^2g^2\sec^2\alpha}{u^2} + 2bg\tan\alpha + u^2\sin^2\alpha$

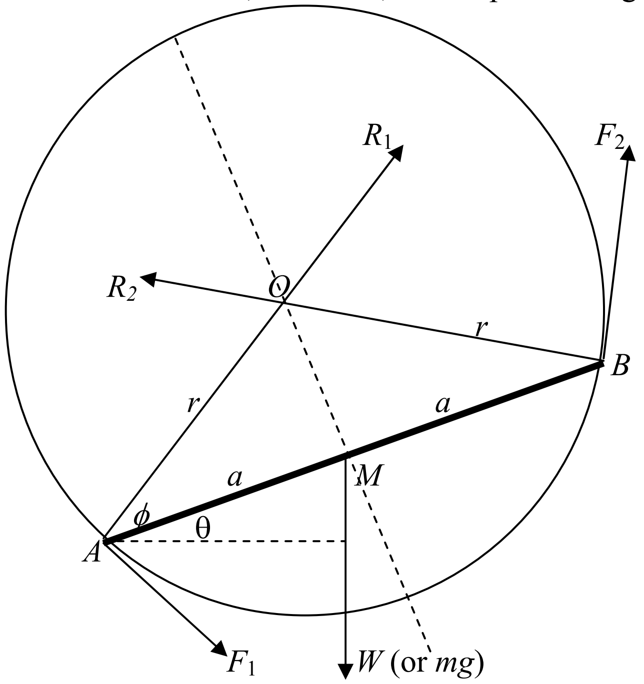
Cancelling $u^2\sin^2\alpha$ both sides & dividing by g $\Rightarrow$ $4h < \dfrac{b^2g\sec^2\alpha}{u^2} + 2b\tan\alpha$

Re-arranging for $\dfrac{1}{u^2}$ $\Rightarrow$ $\dfrac{2(2h - b\tan\alpha)}{b^2g\sec^2\alpha} < \dfrac{1}{u^2}$

Using the first result, $\dfrac{1}{u^2} < \dfrac{2(h - a\tan\alpha)}{a^2g\sec^2\alpha}$, in here $\Rightarrow$ $\dfrac{2(2h - b\tan\alpha)}{b^2g\sec^2\alpha} < \dfrac{2(h - a\tan\alpha)}{a^2g\sec^2\alpha}$

Re-arranging for $\tan\alpha$ $\Rightarrow$ $ab(b - a)\tan\alpha < h(b^2 - 2a^2)$, which leads to the required final answer
$\tan\alpha < \dfrac{h(b^2 - 2a^2)}{ab(b - a)}$. However, it is necessary (since we might otherwise be dividing by a quantity that could be negative) to explain that $b > a$ (we are now on the other side of the net to the projection point) else the direction of the inequality would reverse.

Question 10
As with many statics problems, a good diagram is essential to successful progress. Then there are relatively few mechanical principles to be applied ... resolving (twice), taking moments, and the standard "Friction Law". It is, of course, also important to get the angles right.



Taking moments about M :
$R_1\,a\sin\phi = R_2\,a\sin\phi + F_1\,a\cos\phi + F_2\,a\cos\phi$

Using the Friction Law : $F_1 = \mu R_1$ and $F_2 = \mu R_2$

Dividing by $\cos\phi$ and re-arranging
$R_1\tan\phi = R_2\tan\phi + \mu R_1 + \mu R_2$
$\Rightarrow (R_1 - R_2)\tan\phi = \mu(R_1 + R_2)$

For the second part, it seems likely that we will have to resolve twice (not having yet used this particular set of tools), though we could take moments about some other point in place of one resolution. There is also the question of which directions to resolve in – here, it should be clear very quickly that "horizontally and vertically" will only yield some very messy results.

Moments about O : $\mu(R_1 - R_2)\,r = W\,r\sin\phi\sin\theta$
Resolving // AB : $(R_1 - R_2)\cos\phi + \mu(R_1 + R_2)\sin\phi = W\sin\theta$
(Give one **A1** here if all correct apart from a – sign)
Resolving $\perp^{r}$ AB : $(R_1 + R_2)\sin\phi - \mu(R_1 - R_2)\cos\phi = W\cos\theta$
Note that only two of these are actually required, but it may be easier to write them all down first and *then* decide which two are best used.

Dividing these last two eqns. $\Rightarrow\ \tan\theta = \dfrac{(R_1 - R_2)\cos\phi + \mu(R_1 + R_2)\sin\phi}{(R_1 + R_2)\sin\phi - \mu(R_1 - R_2)\cos\phi}$

Using first result, $\mu(R_1 + R_2) = (R_1 - R_2)\tan\phi\ \Rightarrow\ \tan\theta = \dfrac{(R_1 - R_2)\cos\phi + (R_1 - R_2)\tan\phi\sin\phi}{(R_1 - R_2)\dfrac{\tan\phi}{\mu}\sin\phi - \mu(R_1 - R_2)\cos\phi}$

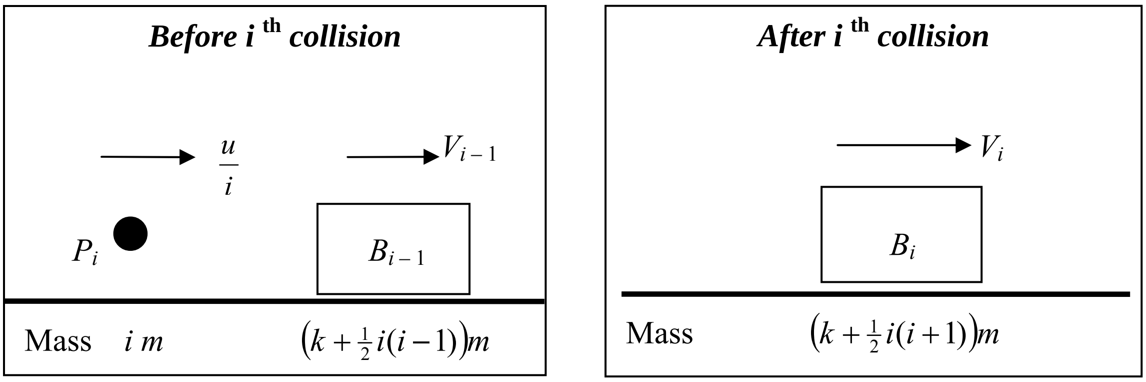
$\Rightarrow\ \tan\theta = \dfrac{\cos\phi + \tan\phi\sin\phi}{\dfrac{\tan\phi}{\mu}\sin\phi - \mu\cos\phi}$. (There is no need to note that $R_1 \neq R_2$ for then the rod would hve to be positioned symmetrically in the cylinder.)

Multiplying throughout by $\mu\cos\phi\ \Rightarrow\ \tan\theta = \dfrac{\mu(\cos^2\phi + \sin^2\phi)}{\sin^2\phi - \mu^2\cos^2\phi} = \dfrac{\mu}{1 - \cos^2\phi - \mu^2\cos^2\phi}$ and, using

$\cos\phi = \dfrac{a}{r}$ gives $\tan\theta = \dfrac{\mu}{1 - \dfrac{a^2}{r^2} - \mu^2\left(\dfrac{a^2}{r^2}\right)} = \dfrac{\mu r^2}{r^2 - a^2(1 + \mu^2)}$.

Finally, $\tan\lambda = \mu = \left(\dfrac{R_1 - R_2}{R_1 + R_2}\right)\tan\phi$, from the first result, $< \tan\phi\ \Rightarrow\ \lambda < \phi$.

Question 11
Again, a diagram is really useful for helping put ones thoughts in order; also, we are going to have to consider what is going on generally (and not just "pattern-spot" our way up the line).



Using the principle of *Conservation of Linear Momentum*,
CLM → $\quad m\,u + M\,V_{i-1} = (M + im)\,V_i \qquad$ (NB $V_0 = 0$) leads to

$$V_1 = \frac{u}{k+1},\ V_2 = \frac{2u}{k+1+2},\ V_3 = \frac{3u}{k+1+2+3},\ \ldots,\ V_n = \frac{nu}{k+\frac{1}{2}n\,(n+1)} = \frac{2nu}{2k+n\,(n+1)}.$$

Alternatively, CLM → for *all* particles gives $mu + 2m\left(\frac{u}{2}\right) + 3m\left(\frac{u}{3}\right) + \ldots\ldots + nm\left(\frac{u}{n}\right) = \left(k + \frac{1}{2}n(n+1)\right)mV$,

and rearranging for $V = V_n$ yields $V_n = \dfrac{2nu}{2k + n\,(n+1)}$.

The last collision occurs when $V_n \geq \dfrac{u}{n+1}$, i.e. $\dfrac{2nu}{N(N+1) + n\,(n+1)} \geq \dfrac{u}{n+1}$

$\Rightarrow\ 2n\,(n+1) \geq N(N+1) + n\,(n+1)\ \Rightarrow\ n\,(n+1) \geq N(N+1)\ \Rightarrow$ there are N collisions.

Now, the total KE of all the P_i's is $\displaystyle\sum_{i=1}^{N} \tfrac{1}{2}(i\,m)\left(\frac{u}{i}\right)^2 = \tfrac{1}{2}mu^2 \sum_{i=1}^{N} \frac{1}{i}$.

The final KE of the block is $\tfrac{1}{2}N(N+1)mV_N^{\,2} = \tfrac{1}{2}N(N+1)m\left(\dfrac{u}{N+1}\right)^2 = \tfrac{1}{2}mu^2\left(\dfrac{N}{N+1}\right)$.

Therefore, the loss in KE is the difference: $\displaystyle \tfrac{1}{2}mu^2 \sum_{i=1}^{N}\frac{1}{i} - \tfrac{1}{2}mu^2\left(\frac{N}{N+1}\right)$.

Since $\dfrac{N}{N+1} = 1 - \dfrac{1}{N+1}$, the loss in KE is $\displaystyle \tfrac{1}{2}mu^2\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{N} - 1 + \frac{1}{N+1}\right) = \tfrac{1}{2}mu^2 \sum_{i=2}^{N+1}\left(\frac{1}{i}\right)$.

Question 12
This *can* be broken down into more (four) separate cases, but there is no need to:
P(light on) $= p \times \frac{3}{4} \times \frac{1}{2} + (1-p) \times \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}(1+2p)$, and then the conditional probability
P(Hall | on) $= \dfrac{\frac{1}{8}(1-p)}{\frac{1}{8}(1+2p)} = \dfrac{(1-p)}{(1+2p)}$.

To make progress with this next part of the question, it is important to recognise the underlying binomial distribution, and that each day represents one such (Bernouilli) trial. We are thus dealing with B(7, p_1), where $p_1 = \dfrac{(1-p)}{(1+2p)}$ is the previously given answer.

For the modal value to be 3, we must have P(2) < P(3) < P(4); that is,

$$\binom{7}{2}(p_1)^2(1-p_1)^5 < \binom{7}{3}(p_1)^3(1-p_1)^4 \quad \textit{and} \quad \binom{7}{4}(p_1)^4(1-p_1)^3 < \binom{7}{3}(p_1)^3(1-p_1)^4 .$$

Using $p_1 = \dfrac{(1-p)}{(1+2p)}$ gives

$$21\left(\frac{1-p}{1+2p}\right)^2\left(\frac{3p}{1+2p}\right)^5 < 35\left(\frac{1-p}{1+2p}\right)^3\left(\frac{3p}{1+2p}\right)^4 \Rightarrow 3(3p) < 5(1-p) \Rightarrow p < \tfrac{5}{14}$$

and

$$35\left(\frac{1-p}{1+2p}\right)^4\left(\frac{3p}{1+2p}\right)^3 < 35\left(\frac{1-p}{1+2p}\right)^3\left(\frac{3p}{1+2p}\right)^4 \Rightarrow (1-p) < (3p) \Rightarrow p > \tfrac{1}{4} .$$

Question 13
Working with the distribution $P_o(\lambda = k\pi y^2)$, P(no supermarkets) = $e^{-k\pi y^2}$ and $P(Y < y) = 1 - e^{-k\pi y^2}$.
Differentiating w.r.t. y to find the pdf of $Y \Rightarrow f(y) = 2k\pi\, y\, e^{-k\pi y^2}$, as given. Then
$E(Y) = \int_0^\infty 2k\pi\, y^2\, e^{-k\pi y^2}\, dy$. Using *Integration by Parts* and writing $2k\pi\, y^2\, e^{-k\pi y^2}$ as $y\left(2k\pi\, y\, e^{-k\pi y^2}\right)$

gives $E(Y) = \left[y\left(e^{-k\pi y^2}\right)\right]_0^\infty + \int_0^\infty e^{-k\pi y^2}\, dy = 0 + \int_0^\infty e^{-k\pi y^2}\, dy$. It is useful (but not essential) to use the

simplifying substitution $x = y\sqrt{2k\pi}$ at this stage to get $\dfrac{1}{\sqrt{2k\pi}}\int_0^\infty e^{-\frac{1}{2}x^2}\, dx = \dfrac{1}{\sqrt{2k\pi}}\sqrt{\dfrac{\pi}{2}} = \dfrac{1}{2\sqrt{k}}$ (by the given result, relating to the standard normal distribution's pdf, at the very beginning of the question).

Next, $E(Y^2) = \int_0^\infty 2k\pi\, y^3\, e^{-k\pi y^2}\, dy$, and using *Integration by Parts* and, in a similar way to earlier,

writing $2k\pi\, y^3\, e^{-k\pi y^2}$ as $y^2\left(2k\pi\, y\, e^{-k\pi y^2}\right)$, $E(Y^2) = \left[y^2\left(e^{-k\pi y^2}\right)\right]_0^\infty + \int_0^\infty 2y\, e^{-k\pi y^2}\, dy$

$= 0 + \dfrac{1}{k\pi}\int_0^\infty 2k\pi\, y\, e^{-k\pi y^2} = \dfrac{-1}{k\pi}\left[e^{-k\pi y^2}\right]_0^\infty$ (using a previous result, or by substitution) $= \dfrac{1}{k\pi}$

$\Rightarrow \text{Var}(Y) = \dfrac{1}{k\pi} - \dfrac{1}{4k} = \dfrac{4-\pi}{4k\pi}$, the given answer, as required.